LOGIC for PRECALCULUS

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I. INTRODUCTION

We're going to learn some logical methods that will help us in math.

The methods that we'll study deal with

- understanding logical symbolism,
- evaluating the truth of compound statements,
- establishing and utilizing logical equivalences,
- validating arguments, and
- proving elementary theorems.

But we have to work up to it.

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SYMBOLS

Well thought-out and carefully defined symbols are important in modern mathematical communication, and your understanding of the symbols of math and logic is essential for clear reasoning, problem solving and the communication of such results.

I think that you'll agree that math is easier to "do" using symbols rather than lots of words. For example, consider

$$5\left(x+3\right)=5x+15$$

vs.

> "Five times an unknown quantity which has been increased by three is fifteen more than five times that unknown quantity."[1]

Which one seems easier to comprehend?

Analogously, it is easier to see the structure of, and deal with, mathematical and logical operations when they have been put into symbolic form.[2]

However, the symbols themselves, which have been created to facilitate and enhance the "doing" of math and logic, have created a body of material requiring its own special learning curve. In effect, the symbols

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[1] Something along these lines is how the "ancient Egyptians" stated this sort of thing.
[2] Both cases assume, of course, that we have a correct understanding of the meaning and use of the symbols involved.

have become a subject which must be mastered in order to get to the science itself.[3] So, we must study the "Science of Math and/or Logic" (the Subject) and the "Science of the Symbols of Math and Logic" (the Symbols).

We must ever keep clear in our minds that these are two separate (but intimately related) subjects, and we must never make the mistake of believing that

- a mastery of the Symbols equates with a mastery of the Subject, or
- a mastery of the Symbols is unnecessary for a mastery of the Subject.

As we shall learn to say, "While a mastery of the Symbols is a ***necessary condition*** for a mastery of the Subject, it is not a ***sufficient condition*** for a mastery of the Subject."

Moreover, the improper use of symbols creates an area in which reasoning itself is obscured and students often become frustrated.

For example, entering

2+3/5

into your calculator when you mean to enter

(2+3)/5

represents an incomplete understanding of the parentheses symbols **"("** and **")"**.

Without the requisite understanding of the symbols, the very utility for which they were created may be lost.

THEOREMS

Also, an extremely important consideration in problem solving is the understanding and application of **mathematical theorems**.

What is a theorem? Here is one definition. A theorem is

> "… a formula, proposition, or statement in mathematics or logic deduced or to be deduced from other formulas or propositions…."[4]

Why bother with theorems? The utility of recording proven facts as theorems and using these theorems to establish yet other facts has been utilized since (at least) the time of Euclid. There is a quotation from Descartes of which I am particularly fond. In his *Discours de la méthode*, as he describes his own method for research, Descartes says:

> ". . . each truth discovered was a rule available in the discovery of subsequent ones." [5]

What do you do with theorems? Two things. You prove theorems and you use theorems. So you need to know how to do both things. But first you might ask "Why?" "Why do either?"

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[3] Here I use the term "science" to stand for "mathematics or logic;" however, my remarks here apply just as well to other sciences, such as physics and chemistry.

[4] Merriam-Webster Online Dictionary copyright **© 2006-2007 Merriam-Webster, Incorporated**. http://mw1.merriam-webster.com/dictionary/theorem

[5] http://www.literature.org/authors/descartes-rene/reason-discourse/chapter-02.html. See the 12th paragraph.

- **Why prove theorems?** Well a theorem provides you with a mathematical rule, or tool, which can be used in problem solving on in proving other theorems, which in turn provide more problem-solving tools. And logically proving a theorem establishes the theorem's validity – that it is a good tool; that it will provide correct, reliable results (if properly applied). In many cases, a proof also provides some insight into math techniques which often come into play in circumstances where the theorem has application, so the proof can provide you with a template or model of how to attack some particular problem. Finally, the practice of proving theorems sharpens our analytical skills and helps us appreciate the need for both good reasoning to be applied in all scientific endeavor and good reasons to be given in all rational arguments.

- **Why use theorems?** Theorems are tools which facilitate problem solving.

- **How do you prove theorems?** That's the subject of entire books. We'll be doing some theorem proving in the next few class periods, and you will undoubtedly be doing more in future math and engineering classes. As we do proofs, you'll see some of the standard techniques.

- **How do you apply theorems?** One point in the studying all this material about logic is to help you understand how to apply theorems. It's much easier to show you how to apply theorems than it is to explain how to apply theorems – So let's just wait a bit until we've built up a foundation and then we'll apply some theorems.

What is the "converse" of a theorem? When teachers talk about theorems they often talk about the "converse" of the theorem. We need to ask "What is a 'converse?'" and "What is its relationship to the theorem?"

Many times a theorem will be stated in the form of a conditional statement (an "if…, then …." statement).

For example,

"If $x = 2$, then $x^2 = 4$."

This is a true statement (in effect a *small* theorem) which is frequently used in problem solving; however, its **converse**[6], which is

"If $x^2 = 4$, then $x = 2$,"

and which is quite often misused in problem solving, is not a true statement.

So, if a theorem is in the form

"If P, then Q," which is $P \Rightarrow Q$ in symbols.

its converse is in the form

"If Q, then P." or $Q \Rightarrow P$

For another example of a theorem (which is true) and its converse (which in this case is not true), let us assume that we are confining our discussion to numbers greater than 2. With this stipulation, the following is true:

[6] We'll discuss the **converse** later in these notes.

"If p is a prime number, then p is an odd number."[7]

The converse of this conditional statement (theorem) is:

"If p is an odd number, then p is a prime number."

And this is, of course, a false statement.

Now, don't get me wrong! Sometimes a conditional statement and its converse are both true. For example:

"If a is less than b, then b is greater than a."

The converse of this statement is:

"If b is greater than a, then a is less than b."

In this example both statements are true.

The important point is that if I am given a true conditional statement, there is no way of knowing ahead of time whether its converse is true or false.

The (false) belief that a converse is true just because the conditional statement is true is one of the biggest mistakes made by students of math. And that's why we mention it early in our discussion!

It is important that we begin our study of logic with an investigation of the symbols and their proper usage – that is, we begin with a study of the **syntax of logic**.

We'll also need some definitions.

II. DEFINITIONS

Def. 1: **Statement**: A **statement** is a sentence[8] that is either true (**T**) or false (**F**), but not both simultaneously.

Example[9]: (a) Doug Jones is 6 ft. 9 in. tall.

(b) $2 \times 3 = 6$

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[7] As you may recall, 2 is the only even prime number.
[8] This could be an English sentence, a German sentence, a Chinese sentence, or a mathematical sentence. And recall that a mathematical sentence is an **equation**, or an **inequality**, or the like.

[9] Statement (a) has a **truth value** of **F**, and statement (b) has a **truth value** of **T**.

Def. 2: **Open Sentence**: An **open sentence** is a sentence with a **variable** in it such that for some (perhaps none) of the substitution values of the variable the sentence becomes a **true statement** and for all other substitutions the sentence becomes a **false statement.**

Example: $3x^2 + 2x - 5 = 0$ is an open sentence.

Def. 3: **Domain of a Variable**: The **domain of a variable in an open sentence** is the set of all possible substitutions which "make sense" when substituted into the open sentence.

Def. 4: **Solution Set**: The **solution set** of an **open sentence** is the set of all values of the **domain of the variable** such that the resulting substitution yields a **true statement**.

Examples: **(a)** "He was 42nd President of the USA." [10]

Analysis:

(i) Variable: "He"

(ii) Domain: $Dom = \{ x \mid x \ is \ a \ man \ born \ in \ the \ US \ before \ 1958. \}$ [11]

(iii) Several Possible Substitutions:

"Yogi Berra was 42nd President of the USA."

"Bruce Springsteen was 42nd President of the USA."

"George Herbert Walker Bush was 42nd President of the USA."

"Bill Clinton was 42nd President of the USA."

(iv) Solution Set: The solution set may be written as either

$\boxed{\{ \text{ Bill Clinton } \}}$ or $\boxed{\{ \ x \mid x \text{ is Bill Clinton } \}}$

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(b) $x^2 - 3 = 1$

Analysis:

(i) Variable: "x"

[10] This is an **open sentence**; as it stands, it is neither **true** nor **false**, but when you *substitute* a specific name for "he," the sentence *becomes* either **true** or **false**.

[11] Section 1 of Article 2 of the U.S. Constitution states that a President must be a natural born citizen of the united States, be at least 35 years old, and have lived in the U.S. for at least 14 years. Also, the 42nd President of the USA was elected in 1992. And 57+35 = 92.

(ii) Domain:[12] $\mathbb{R}$

(iii) Several Possible Subs.

$(15)^2 - 3 = 1$ . . . . (**F**)

$(-2)^2 - 3 = 1$ . . . . (**T**)

(iv) Solution Set:

$\{\ -2,\ 2\ \}$ or $\{\ x \mid x = -2\ \ or\ \ x = 2\ \}$

III. PROBLEMS

A. Identify each of the following as (a) True Statement; (b) False Statement; (c) Statement (Truth Value Unknown); (d) Open Sentence; (e) None of These.

1. _____ Eat your spinach.

2. _____ It rained 37.1 inches in Bombay, India, in one day.

3. _____ She was senior class president of Coral Gables High School (Miami, FL) in 1960.

4. _____ $7 + 5 = 2$

5. _____ $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

6. _____ $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Answers: 1. e 2. a or c 3. d 4. b 5. e 6. d

7. _____ The day after Tuesday is Wednesday.

8. _____ Tiger Woods is a professional tennis player.

9. _____ Every triangle is hungry.

10. _____ There is life on Titan.

11. _____ $x^2 + x + 2 = 0$

12. _____ $x^2 + 9 = (x+3)(x-3)$

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[12] Here, of course, $\mathbb{R}$ is the set of **real numbers**. By the way, do you know exactly what a “real number” is?

13. _____ What's up?

14. _____ Don't run with scissors in your hands.

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B. Find the solution set. Express your answer in either interval notation, roster notation, or set-builder notation – whichever seems most appropriate in each case.

1. $x^2 + 4x - 4 = 0$

2. $|\ x - 4\ | < \frac{1}{10}$

3. Members of the Gold Medal 2004 Women's US Olympic Beach Volleyball Team.

4. $\frac{1}{1-\sqrt{x}} = \frac{1+\sqrt{x}}{1-x}$

<u>Answers</u>: 1. $\left\{\ 2-2\sqrt{2},\ 2+2\sqrt{2}\ \right\}$; 2. $(3.9,\ 4.1)$ or $\left\{x \mid 3.9 < x < 4.1\ \right\}$;

3. $\{$ Kerri Walsh, Misty May $\}$ 4. $[0, 1) \cup (1, \infty)$

5. $3x = 8$

6. $3x^2 = 27$

7. $x^2 - 4x + 4 = 0$

8. $x^2 + 121 = 0$

9. $x^2 \geq 0$

10. $x = \frac{1}{x}$

11. $\{x \mid x \textit{ is a prime number and } x \textit{ is even.}\}$

12. $x < 0 \textit{ and } x^2 = 36$.

13. $|\ x + 3\ | < \frac{1}{5}$

IV. COMPOUND STATEMENTS

Def. 5: A **simple statement** is a statement containing just one idea or concept and no part of which conveys the idea or concept of the entire statement.

Example: "My car is blue." is a simple statement, because none of the pieces of the statement, such as "my," "car," "is," "blue," "my car," "car blue," *etc,* convey the idea of the entire statement.

Def. 6: A **compound statement** is a statement in which one or more simple statements are combined using one or more **logical connectives**.

These **logical connectives** are listed below, and their use is described at length in the material that follows.

The **logical connectives** we use to compound statements are

Name of Operation	Symbol	Name of Symbol	Symbol Pronounced
Negation	$\sim$	tilde	not
Disjunction	$\vee$	wedge	or
Conjunction	$\wedge$	and	and
Material Implication	$\Rightarrow$	arrow	implies
Material Equivalence	$\Leftrightarrow$	double-headed arrow	Is equivalent to

Negation is a **unary operation**; the tilde operates on one statement (simple or compound).

Thus, if p is defined [13] as *p: It is raining.*

Then $\sim p: \ It\ is\ not\ raining.$

The other four logical connectives are **binary operators**; that is, each combines two statements:

Thus if *p: It is raining; q: I see a rainbow.*

Then $(p \wedge q)$: *It is raining, and I see a rainbow.*

$(p \vee q)$: *Either it is raining, or I see a rainbow.*

$(p \Rightarrow q)$: *If it is raining, then I see a rainbow.*

$(p \Leftrightarrow q)$: *It is raining if and only if I see a rainbow.*

[13] We'll use lower-case letters to stand for statements, and we'll use the colon ":" to mean "… is defined as …."

We also use **parentheses** to group compound statements into doubly and triply compound statements, such as

$\left(\left(p \Rightarrow q\right) \wedge p\right) \Rightarrow q$: *If whenever it is raining I see a rainbow and it is in fact raining, then I do see a rainbow.*

V. TRANSLATIONS or TRANSCRIPTIONS

The first thing that we need to practice here is the **translation** of compound statements from words to symbols and *vice-versa*.

As you have seen above, we use letters to stand for statements. In these notes we'll generally use lower-case letters to stand for simple statements. And we'll use parentheses to group the letters and connectives in the order of application.

In a given context, the list of letters we use to stand for statements is called a **dictionary**.

Suppose that for these examples we define the following Dictionary "A:"

p: It is raining.
q: Doug gets wet.
r: The sun is shining.

Then

(i) "Either it is not raining, or Doug gets wet." $\xrightarrow{\text{translates}}$ $\left(\sim p\right) \vee q$, but we usually[14] write $\sim p \vee q$.

(ii)[15] "Neither is it raining nor does Doug get wet." $\xrightarrow{\text{translates}}$ $\sim\left(p \vee q\right)$

(iii) "If the sun is shining, then it is not raining." $\xrightarrow{\text{translates}}$ $r \Rightarrow \left(\sim p\right)$, but we usually write $r \Rightarrow \sim p$

(iv)[16] $\left(r \wedge q\right) \Rightarrow p$ $\xrightarrow{\text{translates}}$ "If the sun is shining and Doug gets wet, then it is raining."

(v) $r \wedge \left(q \Rightarrow p\right)$ $\xrightarrow{\text{translates}}$ "The sun is shining, and if Doug gets wet, it is raining."

So, in logic, as in math, ***parentheses do make a difference***!

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Often, instead of using the boring *p* and *q*, we use letters in our dictionary that remind us of the statement represented:

[14] In this case the omission of the parentheses causes no ambiguity; thus, it is the usual case to omit the parentheses.
[15] But in this case the parentheses are absolutely necessary.
[16] Also here the parentheses are absolutely necessary to explicitly define the order of operations.

Dictionary "B:"

a: The apple is ripe.
b: The bird eats the apple.
c: The cat stalks the bird.
d: The dog chases the cat.
r: It rains.

(vi) $\left(\left(a \vee b\right) \wedge c\right) \Rightarrow d \xrightarrow{\text{translates}}$ "If both the apple is ripe or the bird eats it and the cat stalks the bird, then the dog chases the cat."

(vii) $\left(a \vee \left(b \wedge c\right)\right) \Rightarrow d \xrightarrow{\text{translates}}$ "If either the apple is ripe or both the bird eats the apple and the cat stalks the bird, then the dog chases the cat."

(viii) $\sim\left(c \wedge d\right) \xrightarrow{\text{translates}}$ "It is not the case that both[17] the cat stalks the bird, and the dog chases the cat."

(ix) $\sim c \wedge d \xrightarrow{\text{translates}}$ "It is not the case that the cat stalks the bird, and the dog chases the cat." OR

"The cat doesn't stalk the bird, and the dog does chase the cat."

(x) $\sim c \vee \sim d \xrightarrow{\text{translates}}$ "Either the cat doesn't stalk the bird, or the dog doesn't chase the cat.[18]"

VI. PROBLEMS

(A) Translate from English into symbols. Use the letters given to stand for affirmative concepts.[19]

1. Either you get the car or Bob gets the boat. (c, b)
2. You will get a Mustang for your birthday, and if Jeremy is good, he will get a Honda. (m, g, h)
3. [[*** Let's make up one in class! ***]]
4. If you don't finish your homework, then you can't go out tonight. (f, g)[20]

[17] Please carefully compare this statement and the next to note the difference in meaning that the word "both" makes.
[18] As we shall see, example (viii) and example (x) ***mean the same thing***. This is important!
[19] If you are "on your own," that is, if you are not given specific letters to use, then you may need to make a **dictionary**.
[20] Here is a **Dictionary** for this problem: f: You finish your homework.
g: You do go out tonight.
Often, but not always, we state dictionary definitions in the affirmative.

5. If you clean-up your room, then you can go to Europe. (c, g)

Answers: 1. $c \vee b$ 2. $m \wedge (g \Rightarrow h)$ 3. __________ 4. $\sim f \Rightarrow \sim g$
5. $c \Rightarrow g$

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6. If either Joe or Bob goes to the party, then Anna will not go. (j, b, a)

7. Whenever the sky is blue and the clouds are white, it implies that it will rain in the evening or the wind will blow. (s, c, r, w)

8. Paul passes and Frank fails the test if and only if Eldonna exempts the test.

(B) Translate from symbols into English,[21] given this **dictionary**:

d: The night was dark. *w: The cabin was warm.* *b: The moon was bright.*
s: The night was stormy. *h: The coffee was hot.* *e: I went to bed early.*

Example: $(d \wedge s) \wedge w \xrightarrow{\text{translates}}$ "The night was dark and stormy, but[22] the cabin was warm."

Problems:

1. $s \vee b$ 2. $s \Rightarrow \sim b$ 3. $(\sim h \vee (s \wedge \sim w)) \Rightarrow e$

Answers: 1. Either the night was stormy or the moon was bright.

2. If the night was stormy, then the moon was not bright.

3. If either the coffee was not hot, or both the night was stormy and the cabin was not warm, then I went to bed early.

4. $(\sim h \vee d) \wedge e$ 5. $b \wedge \sim d$ 6. $\sim h \vee (d \wedge e)$ 7. $\sim (h \vee d) \wedge e$

VII. MORE ON ORDER OF OPERATIONS

Conjunction ("and"), disjunction ("or"), material implication ("implies"), and material equivalence ("if and only if") are **binary operators**, which means that each operates on *two statements*. On the other hand, negation ("not") is a **unary operator**; it operates on *one statement*.

[21] I'm using the word "English" in a generic way. I really mean "Native Language," be it Spanish, German, Chinese, *etc*.
[22] "But" really means "and," usually with an opposite "spin."

Thus statements must be grouped, using **parentheses**, in such a way that the compound expressions of letters (statements) and operators make sense.

Here are two examples –

(i) $p \wedge q \Rightarrow r$ is ***ambiguous*** – it has more than one possible meaning. Its author *could have meant* $p \wedge (q \Rightarrow r)$, or the author *could have meant* $(p \wedge q) \Rightarrow r$, but we don't know which. And there is a difference! [23]

(ii) $p \wedge \Rightarrow r$ is ***nonsense!*** It means nothing.

Neither one is a **well-formed formula**, a **wff**.

A properly written symbolic expression[24] is called a **well-formed formula**, a **wff** for short. In calculating the **truth values** of compound statements, we require **wff**s! So it is important that you be able to recognize **wff**s and **non-wff**s.

VIII. PROBLEMS

Which are **wff**s? Answer **WFF** or **XXX**.

1. _____ $a \wedge (b \Rightarrow c)$
2. _____ $p \Rightarrow (q \Leftrightarrow r) \vee s$
3. _____ $p \wedge \vee q$
4. _____ $\sim p \Rightarrow \sim q$

Answers. 1. wff 2. xxx 3. xxx 4. wff

5. _____ $a \Rightarrow \sim b \vee c$
6. _____ $(a \Rightarrow b) \wedge (c \vee \sim d)$
7. _____ $(x \vee \sim y \wedge z) \Rightarrow w$
8. _____ $\sim(\sim s \wedge \sim t)$
9. _____ $(x \vee \sim y) \Rightarrow w$
10. _____ $\sim(p \Rightarrow \sim m) \wedge (\sim a \vee b)$

IX. ANOTHER DEFINITION

Def. 7: A **truth-functional statement** is a compound statement whose truth value can be determined conclusively from the knowledge of the truth values of its component statements.

X. ORDINARY TRUTH TABLES (OTT)

[23] We'll actually soon be able to *prove* this claim.
[24] "Properly written" means with parentheses specifying the order of operations.

One method of determining the truth value of a compound statement is to construct an **ordinary truth table (OTT)** for the statement. In the interest of brevity, an **ordinary truth table** is often called simply a **truth table (TT)**[25]. So, from here on out, we'll call them **truth tables**, and we'll probably just abbreviate them as **TT**.

Note: A **TT** is very similar to the familiar **T-table** used by algebra students in making graphs. The major differences are **(1)** the domains and ranges of functions in algebra class are usually quite large (infinite, in fact), while the domains and ranges of compound statements are very small, and **(2)** the domains and ranges of functions are usually numbers, while the domains and ranges of compound statements are letters, namely **T** and **F**.

XI. THE FIVE BASIC TRUTH TABLES In the following, p and q can represent any two statements, simple or compound.

(A) Negation. (Neg.) "not" – Symbol " ~ "

p	$\sim p$
T	**F**
F	**T**

Thus, if p is true, not-p is false. And if p is false, not-p is true.

(B) Disjunction. (Disj.) "or" – Symbol " $\vee$ "

p	q	$p \vee q$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

The p and the q are called **disjuncts**, and the compound statement $p \vee q$ is called a **disjunction**. A disjunction is **true** if and only if ***at least one of the disjuncts is true***.

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[25] In fact, there is another type truth table, called an **indirect truth table**. But since we are not going to study the indirect truth table in this report, it will not be ambiguous to use the term **truth table** to stand for **ordinary truth table**.

(C) Conjunction. (Conj.) "and" – Symbol " $\wedge$ "

p	q	$p \wedge q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

The p and the q are called **conjuncts**, and the compound statement $p \wedge q$ is called a **conjunction**. A conjunction is **true** if and only if ***both the conjuncts are true***.

(D) Material Implication. (MI) "If…, then …. " – Symbol "$\Rightarrow$ "

p	q	$p \Rightarrow q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

The **material implication** is a **promise**. The "p" is called the **antecedent**, and the "q" is called the **consequent**. The only time material implication is **false** is when the ***promise is broken***; that is, when the antecedent is **true** and the consequence is **false**. For example:

"If you clean up your room, then you can go to Europe." is a promise.

The only way the promise is broken is the case when you *do* clean up your room and you *do not* go to Europe.[26]

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[26] If you do not bother to clean up your room, in other words, if the "p" is **false**, then you haven't even tested the promise. Thus, whether or not you do go to Europe, the promise was not broken, since it was not tested. This is why the last two rows, or scenarios in **(D)** above, result in a **TRUE MI**.

(E) Material Equivalence. (ME) ". . . if and only if . . . . " – Symbol "$\Leftrightarrow$"

p	*q*	$p \Leftrightarrow q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

Material equivalence is **true** when and only when the p and q have the ***same truth value***.[27]

XII. TRUTH VALUES OF MORE COMPLICATED COMPOUND STATEMENTS Suppose we know the truth values of the simple components of a compound **WFF**. How can I quickly determine the truth value of the **WFF**?

The Method of Substitution of Truth Values Suppose that *a, b, c,* and *d* are statements. Further, suppose that we know that *a* is true, *b* is false, *c* is false, and *d* is true. These "givens" are easier to see if we simply write[28]

a: **T** *b*: **F** *c*: **F** *d*: **T**

(i). Example: Now, suppose that we form the compound statement[29]

$$S_1 : (a \Rightarrow b) \vee \sim c$$

How can I determine the truth or falsity of S_1? I **solve the statement** by the **substitution method** – substitute the truth values and simplify the sentence, working from within parentheses[30] and referring to the "five basic truth tables." [31]

Solution.

$$(a \Rightarrow b) \vee \sim c$$
$$(T \Rightarrow F) \vee \sim F$$
$$F \quad \vee T$$
$$\boxed{T}$$

Thus, the compound statement S_1 is **true** for the given set of **truth values** of the constituent parts.

[27] Thus, in line 3 of the **TT** for **ME** in **(E)** above, we see that ***two lies are indeed equivalent!***
[28] And this is how we'll write it from here on.
[29] By the way, S_1 is a **disjunction**, because its **main connective** is the "**wedge**."
[30] Remember that $\sim c$ is really $(\sim c)$.
[31] Which you have "learned by heart!"

(ii) Here is another example:[32] Using the same set of "givens," calculate the **truth value** of

$$S_2 : \left(\sim c \Leftrightarrow (a \wedge \sim b)\right) \vee \sim (d \Rightarrow \sim b)$$

Solution.

$$\left(\sim c \Leftrightarrow (a \wedge \sim b)\right) \vee \sim (d \Rightarrow \sim b)$$
$$\left(\sim F \Leftrightarrow (T \wedge \sim F)\right) \vee \sim (T \Rightarrow \sim F)$$
$$\left(T \Leftrightarrow (T \wedge T)\right) \vee \sim (T \Rightarrow T)$$
$$\left(T \Leftrightarrow T\right) \vee \sim T$$
$$T \vee F$$
$$\boxed{T}$$

Thus under the **hypotheses** given for *a, b, c,* and *d*, the compound statement S_2 is **true**.
Do you see in the solutions above that each successive line is arrived at by referring to the "five basic truth tables?"

XIII. PROBLEMS Given these truth values:

m: **F** *n*: **F** *o*: **F** *p*: **T** *q*: **T** *r*: **T**

Calculate the **truth value** of each of the following compound statements. Also, as a bonus, **identify** the compound statement type (*i.e.* identify the main connective).

1. $(m \wedge \sim n) \wedge (\sim p \Rightarrow q)$ 2. $\sim r \Rightarrow (n \vee \sim n)$ 3. $((m \vee n) \vee r) \Rightarrow (q \Leftrightarrow o)$

Answers: 1. F (conjunction) 2. T (implication[33]) 3. F (implication)

4. $\sim (n \Rightarrow p) \vee \sim (q \Rightarrow \sim r)$ 5. $((p \wedge q) \vee r) \Leftrightarrow ((p \vee r) \wedge (q \vee r))$

6. $((m \Rightarrow p) \wedge m) \Rightarrow p$ 7. $((p \vee m) \wedge \sim p) \Rightarrow m$

XIV. ORDINARY TRUTH TABLES FOR MORE COMPLICATED STATEMENTS Suppose that we have not been given prior knowledge of the truth values of the parts of a compound statement. We can always construct an ordinary truth table, which gives all possible "truth-outcomes" of the compound statement. There are two methods of construction of truth tables in general use. The type of truth table we use is a constructive[34] truth table. That is, we build up the given compound statement in several steps.

---

[32] S_2 is also a **disjunction**.
[33] It is acceptable to say "implication" or "conditional statement" instead of "material implication."
[34] or "progressive"

The steps are –

1. Construction of a "template" **TT** for the given compound statement.
2. Evaluation of the resulting **TT**.
3. Analysis of the Evaluation.

Example:

Let me show you by example. Suppose I wish to construct a truth table for the statement[35]

$$(b \wedge \sim a) \Rightarrow (a \Rightarrow b)$$

Sol.

Step 1 (a): Ask and answer the question: how many distinct **literal variables**[36] are there in the statement? In this case the answer is 2. The number of **lines** in the truth table is given by the formula

$$L = 2^n$$

where n is the number of distinct literal variables in the statement. In this case $n = 2$, thus

$$L = 2^n = 2^2 = 4.$$

Hence, there are four lines in the truth table for the given statement.[37]

Step 1 (b): Start the **TT** with the literal variables[38] in alphabetical order at the top of the **TT** and list all possible "truth permutations." This is **standard form**.[39] I'm going to build my **TT** in several steps just to show you the construction progression in time. In working a problem, you just complete one table.

a	*b*	
T	**T**	
T	**F**	
F	**T**	
F	**F**	
1	2	

It's also a good idea to **number each column** at the bottom. This is the basic set-up. In any two-variable statement, the first column of the **TT** will always be TTFF and the second column will always be TFTF.

Each **row** of a **TT** represents a **scenario** – a possible permutation of **T**'s and **F**'s.

Now let me explain what it is that we're trying to do here –

---

35 This compound statement is an **implication**.
36 In this example the letters "a" and "b" are the **literal variables**.
37 This is a major characteristic of an **ordinary truth table** – it has $L = 2^n$ lines in it.
38 We usually put the literal variables into *alphabetical order* at the beginning of a truth table.
39 Please notice that I set-off the basic literal variables from the remainder of the **TT** by a *double vertical line*.

- We are going to build-up a sequence of columns.
- Each one will have a heading.
- The headings will be the “pieces” of the compound statement.
- The column headings are going to get progressively more complicated.
- Each column heading will be either the negation of a previous column heading or the joining of two previous column headings by one of the four binary connectives.
- The final column heading will be the compound statement we are analyzing.

Step 1 (c): Make a column headings to build-up to the complete compound statement.[40]

a	*b*	$\sim a$	$(b \wedge \sim a)$	$(a \Rightarrow b)$	$(b \wedge \sim a) \Rightarrow (a \Rightarrow b)$
T	**T**				
T	**F**				
F	**T**				
F	**F**				
1	2	3	4	5	6

Now the “template” is set-up for actually evaluating the **TT**.

Step 2 (Evaluation): Now fill in the truth values (either **T** or **F**), referring back to your knowledge of the “five basic truth tables” to determine the truth values in each of these columns.

Please note that the total number of columns in the truth table will be the sum of the number of distinct literal variables plus the total number of connectives occurring in the given statement. Thus, since $(b \wedge \sim a) \Rightarrow (a \Rightarrow b)$ has 2 distinct literal variables and 4 logical connectives ($\wedge, \sim, \Rightarrow, \Rightarrow$) there will be $2+4=6$ total columns in the **TT**.

a	*b*	$\sim a$	$(b \wedge \sim a)$	$(a \Rightarrow b)$	$(b \wedge \sim a) \Rightarrow (a \Rightarrow b)$
T	**T**	**F**	**F**	**T**	**T**
T	**F**	**F**	**F**	**F**	**T**
F	**T**	**T**	**T**	**T**	**T**
F	**F**	**T**	**F**	**T**	**T**
1	2	3	4	5	6

Now that the **TT** is evaluated, we can analyze it.

[40] Let me repeat – In reality you just add to Step 2; you don’t have to start all over. I’m just doing it here to isolate the stages of progress.

Step 3 (Analysis): Let's analyze just what we've got here. In reality, columns 1, 2, and 6 constitute the **TT** for the given statement (which, I remind you, is a **WFF**). Columns 3, 4, and 5 are really just my "work area." But when I am asked to construct an **ordinary truth table** for a given statement, the entire structure as shown in Step 4 is what is expected! For some **TT**s there may be fewer than 6 columns; for some, more than 6.

Look now at what the **TT** tells us

- There are four "scenarios"[41]
  - *a* might be true and *b* might be true;
  - *a* might be true and *b* might be false;
  - *a* might be false and *b* might be true; or
  - *a* might be false and *b* might be false.
- There are no more "truth permutations." That is, there are no more "scenarios" to be considered.
- The **ordinary truth table** gives us the **truth value** of the given compound statement in the *final column* of each scenario.
- Thus, columns 1, 2 and 6 in the example really form the **TT** for the compound statement; columns 3, 4 and 5 are in reality "helper columns" that assist me in finding the truth values listed in column 6.

The above are general facts about **any ordinary truth table**.
Our particular example yields a **rather unusual outcome**; the given statement is **always true**, no matter what the scenario.
Def. 8: A statement that is **always true** is called a **tautology**.[42]

Example: Suppose I consider another statement[43]

$$\sim(p \vee q) \Leftrightarrow (\sim p \ \vee \ \sim q)$$

I'm just going to give you the completed **OTT**, but I think that you can see how I decided to "build" it. [44]

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \vee \sim q$	$\sim(p \vee q) \Leftrightarrow (\sim p \vee \sim q)$
T	**T**	**F**	**F**	**T**	**F**	**F**	**T**
T	**F**	**F**	**T**	**T**	**F**	**T**	**F**
F	**T**	**T**	**F**	**T**	**F**	**T**	**F**
F	**F**	**T**	**T**	**F**	**T**	**T**	**T**
1	2	3	4	5	6	7	8

Thus, you see that the truth value of the statement varies in the four scenarios. The statement is **true** in the first and fourth scenarios, and it is **false** in the second and third scenarios.

[41] Each line in the **truth table** is a "scenario." Each line represents a logical possibility. A **truth table** is in reality a ***possibility chart*** for the given statement.
[42] A **tautology** is also called a **logically true statement**.
[43] This statement is an **equivalence**.
[44] Note: There are 2 distinct literal variables and 6 total connectives; therefore, there are 8 columns in this **TT**.

Def. 9: A statement that is true in some scenarios and false in all others is called **contingent**.

The statement that we just analyzed is a **contingent statement**.

Def. 10: A statement that is always false[45] is called **self-contradictory** or **logically false**.

Example: Let me give you one more example[46]. Construct a **TT** for $(x \wedge y) \wedge \sim x$

Solution.

x	y	$\sim x$	$x \wedge y$	$(x \wedge y) \wedge \sim x$
T	**T**	**F**	**T**	**F**
T	**F**	**F**	**F**	**F**
F	**T**	**T**	**F**	**F**
F	**F**	**T**	**F**	**F**
1	2	3	4	5

Thus, we see from column 5 of the **OTT** that $(x \wedge y) \wedge \sim x$ is **self-contradictory**.

Example: Next, lest you think all **OTT**s have four scenarios, consider[47] $(m \Rightarrow (a \vee d)) \wedge (\sim m \vee \sim d)$

Solution.

Here the **TT** has $L = 2^n = 2^3 = 8$ lines, since the statement contains three distinct literal variables[48].

a	d	m	$\sim d$	$\sim m$	$a \vee d$	$m \Rightarrow (a \vee d)$	$\sim m \vee \sim d$	$(m \Rightarrow (a \vee d)) \wedge (\sim m \vee \sim d)$
T	**T**	**T**	**F**	**F**	**T**	**T**	**F**	**F**
T	**T**	**F**	**F**	**T**	**T**	**T**	**T**	**T**
T	**F**	**T**	**T**	**F**	**T**	**T**	**T**	**T**
T	**F**	**F**	**T**	**T**	**T**	**T**	**T**	**T**
F	**T**	**T**	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**F**	**F**	**T**	**T**	**T**	**T**	**T**
F	**F**	**T**	**T**	**F**	**F**	**F**	**T**	**F**
F	**F**	**F**	**T**	**T**	**F**	**T**	**T**	**T**
1	2	3	4	5	6	7	8	9

[45] This means that the last column of its **truth table** contains ***all Fs.***
[46] This statement is a **conjunction**.
[47] This statement is a **conjunction**.
[48] The **distinct literal variables** are "a," "d," and "m."

Column 9 of this **TT** shows that this statement, a **conjunction,** is **contingent**.

Example: Finally, let's construct an **OTT** for this statement:

$$\left(\sim(p \wedge a) \Leftrightarrow (m \Rightarrow (b \vee c))\right) \vee \left((d \vee a) \Leftrightarrow (b \vee m)\right)$$

First we need to ask ourselves "How many lines does the **TT** have?" Well, there are 7 distinct literal variables, so there will be $L = 2^n = 2^7 = 128$ lines in the **ordinary truth table**. . . .

So, maybe we'll defer this project to a later time!

XV. PROBLEMS **(a)** ***Identify*** the compound statement. **(b)** ***Classify*** the compound statement. **(c)** ***Complete*** the **OTT**.

(i) **Problem:** $((p \Rightarrow q) \wedge p) \Rightarrow q$

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \wedge p$	$((p \Rightarrow q) \wedge p) \Rightarrow q$
T	**T**			
T	**F**			
F	**T**			
F	**F**			
1	2	3	4	5

Answers: **(a)** Name of Statement: **Implication**. **(b)** Classification: **Tautology**
(c) Completed **TT**.

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \wedge p$	$((p \Rightarrow q) \wedge p) \Rightarrow q$
T	**T**	**T**	**T**	**T**
T	**F**	**F**	**F**	**T**
F	**T**	**T**	**F**	**T**
F	**F**	**T**	**F**	**T**
1	2	3	4	5

(ii) <u>Problem</u>: $((a\vee \sim b)\Leftrightarrow b)\wedge(\sim a\Rightarrow b)$

a	b	$\sim a$	$\sim b$	$a\vee \sim b$	$(a\vee \sim b)\Leftrightarrow b$	$\sim a\Rightarrow b$	$((a\vee \sim b)\Leftrightarrow b)\wedge(\sim a\Rightarrow b)$
T	**T**						
T	**F**						
F	**T**						
F	**F**						
1	2	3	4	5	6	7	8

<u>Answers:</u> **(a)** Name of Statement: **Conjunction**. **(b)** Classification: **Contingent**
(c) Completed **TT**.

a	b	$\sim a$	$\sim b$	$a\vee \sim b$	$(a\vee \sim b)\Leftrightarrow b$	$\sim a\Rightarrow b$	$((a\vee \sim b)\Leftrightarrow b)\wedge(\sim a\Rightarrow b)$
T	**T**	**F**	**F**	**T**	**T**	**T**	**T**
T	**F**	**F**	**T**	**T**	**F**	**T**	**F**
F	**T**	**T**	**F**	**F**	**F**	**T**	**F**
F	**F**	**T**	**T**	**T**	**F**	**F**	**F**
1	2	3	4	5	6	7	8

(iii) <u>Problem</u>: $x\wedge \sim x$

x	$\sim x$	$x\wedge \sim x$
T		
F		
1	2	3

<u>Answers:</u> **(a)** Name of Statement: **Conjunction**.
(b) Classification: **Self-Contradictory**
(c) Completed **TT**.

x	$\sim x$	$x\wedge \sim x$
T	**F**	**F**
F	**T**	**F**
1	2	3

(iv) <u>Problem</u>: $\left(\left(p \Rightarrow q\right) \wedge \left(\sim p \Rightarrow s\right)\right) \Rightarrow \left(q \vee s\right)$

(a) Name of statement: ______________________.

(b) Classify this statement _________ as **A**. a tautology. **B**. self-contradictory. **C**. contingent

(c) Complete Its **ordinary truth table**

p	q	s	$\sim p$	$p \Rightarrow q$	$\sim p \Rightarrow s$	$(p \Rightarrow q) \wedge (\sim p \Rightarrow s)$	$q \vee s$	$((p \Rightarrow q) \wedge (\sim p \Rightarrow s)) \Rightarrow (q \vee s)$
T	**T**	**T**						
T	**T**	**F**						
T	**F**	**T**						
T	**F**	**F**						
F	**T**	**T**						
F	**T**	**F**						
F	**F**	**T**						
F	**F**	**F**						
1	2	3	4	5	6	7	8	9

<u>Answers:</u> (a) Implication
(b) A (Tautology)
(c) **TT**.

p	q	s	$\sim p$	$p \Rightarrow q$	$\sim p \Rightarrow s$	$(p \Rightarrow q) \wedge (\sim p \Rightarrow s)$	$q \vee s$	$((p \Rightarrow q) \wedge (\sim p \Rightarrow s)) \Rightarrow (q \vee s)$
T	**T**	**T**	**F**	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**F**	**T**	**T**	**T**	**T**	**T**
T	**F**	**T**	**F**	**F**	**T**	**F**	**T**	**T**
T	**F**	**F**	**F**	**F**	**T**	**F**	**F**	**T**
F	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**
F	**T**	**F**	**T**	**T**	**F**	**F**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**	**T**	**T**
F	**F**	**F**	**T**	**T**	**F**	**F**	**F**	**T**
1	2	3	4	5	6	7	8	9

(v) <u>**Problem**</u>: $((p \vee q) \wedge p) \Rightarrow q$

(a) Name of statement: _______________________.
(b) Classify this statement _________ as **A**. a tautology. **B**. self-contradictory. **C**. contingent
(c) Complete Its **ordinary truth table**

p	*q*	
1	2	

<u>**Answers:**</u> **(a)** Implication
(b) C (Contingent)
(c) **TT**

p	*q*	$p \vee q$	$(p \vee q) \wedge p$	$((p \vee q) \wedge p) \Rightarrow q$
T	**T**	**T**	**T**	**T**
T	**F**	**T**	**T**	**F**
F	**T**	**T**	**F**	**T**
F	**F**	**F**	**F**	**T**
1	2	3	4	5

XVI. DEDUCTIVE ARGUMENTS (VALID)

Def. 11: **Argument –** An **argument** is a set or collection of statements[49]. One of the statements, the **conclusion** is supported by the others, the **premises**. Additionally, there must exist some connection or **linkage** between the **premises** and the **conclusion**.[50]

[49] Such a collection may be more general than a paragraph. I'll normally refer to a collection of sentences that contains an argument as a **passage**.
[50] There are many excellent definitions of "argument." This one, however, is my personal favorite, in that it succinctly captures the salient characteristics of the term. This definition is due to Hurley. (Patrick J. Hurley: *A Concise Introduction to Logic*, 9th Edition ©2006).

The **premises** are the ***reasons*** given in support of the **conclusion**.

The arguer asks the listener to believe or accept some statement as **true**. This statement is the **conclusion**. The listener has the right (and perhaps the duty) to request from the arguer a reason or reasons for believing or accepting the **conclusion** as being **true**. These reasons are the **premises**. Now it's absolutely fundamental that there must be some connection, or **linkage** as it is called in logic, between the **premises** and **conclusion**.

Def. 12: **Deductive Argument**[51] **–** If the **linkage** between **premises** and **conclusion** is ***intended*** to be ***absolute***, so that whenever the **premises** are all **true**, it is ***impossible*** for the **conclusion** to be **false**, then, and only then, is the **argument** said to be **deductive**.

Def. 13: **Valid Argument**[52] **–** If the **linkage** in a deductive argument ***achieves its intended absoluteness***, then, and only then, that argument is **valid**.

Let me give you several more ways to understand a **valid deductive argument**.

- A deductive argument is **valid** if and only if "it is impossible for the conclusion to be false whenever all the premises are true."
- "**valid**" means "if[53] all the premises are true, then the conclusion absolutely, positively must be true."
- "**valid**" means that the statement "The conjunction of the premises implies the conclusion" is a tautology.

This latter restatement of **validity** provides us with a straightforward, mechanical way of testing the validity of a deductive argument —

- Form the conjunction of the premises;
- Put it in parentheses;
- "Arrow" it to the conclusion.
- Construct an **ordinary truth table** for the resulting statement.[54]
- If the **truth table** shows that the **implication** is a **tautology**, then the argument is **valid**. However, if the **implication** is **not a tautology**, then the argument is **invalid**.

Def. 14: **Sound Argument**[55] **–** If a **valid deductive argument** has all **true premises**, then, and only then, that argument is **sound**.

I think that before we go any further, we need some concrete examples.

---

[51] There is another, very important type of argument – the **inductive argument**. We shall not study inductive arguments in this report; however, for your information I'll give you the basic definitions associated with **inductive arguments**.
Def. 12a: **Inductive Argument –** If the **linkage** between **premises** and **conclusion** is ***intended*** to be ***at best probable***, so that whenever the **premises** are all **true**, it is ***probable*** that the **conclusion** is **true**, then, and only then, is the **argument** said to be **inductive**.

[52] Def. 13a: **Strong Argument –** The **strength** of an inductive argument ***is directly proportional to the probability of the linkage***; the greater the probability, the **stronger** the argument.

[53] **IMPORTANT**: "Valid" does NOT mean that the premises are true! It means that IF the premises were true, then the conclusion would have to be true.

[54] Note — This "resulting statement" is an **implication**.

[55] Def. 14a: **Cogent Argument –** If a **strong inductive argument** has all **true premises**, then, and only then, that argument is **cogent**.

<u>XVII. EXAMPLE</u> Here is a very simple argument in paragraph form.

"A number is either odd or even. This number is not odd; therefore, it is even."

Analysis:

1. Read the passage to determine (or confirm) that it does contain an argument.
2. If you suspect that the passage does contain an argument, then
   a. determine the **conclusion**[56]
   b. determine the **premises**[57]
3. The next step really depends upon how complicated the argument is and exactly how the instructions read.
   a. You may want to (or need to) rewrite the argument in English sentences[58], but in **vertical order**, like this:

      The number is odd or it is even.
      It is not odd.
      The number is even.

   b. You may want to first construct a **dictionary**, like this:

      o: The number is odd.
      e: The number is even.

   c. Then write the argument symbolically in **vertical order**, like this:

$$\begin{array}{l} o \vee e \\ \underline{\sim o} \\ e \end{array}$$

This is **standard symbolic vertical order.**

The point of step 3 is to end up in standard symbolic vertical order (we'll simply say **standard order** from now on) in such a way that the "reader" will not be confused by the meaning of your letters.

4. Rewrite the argument as an **implication**: $\left(\left(o \vee e\right) \wedge \sim o\right) \Rightarrow e$
5. Construct a **TT** for this **implication**:

e	o	$\sim o$	$o \vee e$	$\left(o \vee e\right) \wedge \sim o$	$\left(\left(o \vee e\right) \wedge \sim o\right) \Rightarrow e$
T	**T**	**F**	**T**	**F**	**T**
T	**F**	**T**	**T**	**T**	**T**
F	**T**	**F**	**T**	**F**	**T**
F	**F**	**T**	**F**	**F**	**T**
1	2	3	4	5	6

[56] Remember, the **conclusion** is what the argument is trying to "sell" you.
[57] Remember, the **premises** are the *reasons* that you should "buy" the **conclusion**.
[58] It is permissible to rephrase the statements (both premises and conclusion) as necessary to render the underlying **propositions** in sentence form.

6. The argument is **valid** if and only if the **implication** is a **tautology**. **<u>This</u> implication** is a **tautology**; therefore, **<u>this argument is valid</u>**.

Now it would really be a pain in the neck if we had to check out the validity of every argument we encountered using the above process. But we don't! The really nice thing about **standard form** is that it allows you to see argument patterns! If a pattern (or **argument form**, as it is officially called) proves to be **valid**, then every particular argument (**substitution instance**) that has that pattern is a valid argument.

However, I think that we first need some practice validating arguments.

XVIII. PROBLEMS (VALIDATING ARGUMENTS)

I'll give you the argument form and you validate it. First, write the **implication**. Second, complete the **TT** for the **implication**. Third, interpret your results in sentence form.

I'll do the first one for you.

(i) Given:

$$p \Rightarrow q$$
$$\underline{p\quad\quad}$$
$$q$$

Sol. **(1)** The implication is $\left(\left(p \Rightarrow q\right) \wedge p\right) \Rightarrow q$

(2) The truth table is

P	q	$p \Rightarrow q$	$(p \Rightarrow q) \wedge p$	$\left((p \Rightarrow q) \wedge p\right) \Rightarrow q$
T	**T**	**T**	**T**	**T**
T	**F**	**F**	**F**	**T**
F	**T**	**T**	**F**	**T**
F	**F**	**T**	**F**	**T**
1	2	3	4	5

(3) Interpretation: Since the final column is all **T**s, the implication is a tautology. Therefore, the argument is valid.

Validate these arguments:

(ii) Given:

$$\begin{array}{l} p \Rightarrow q \\ \underline{p \vee q} \\ q \Rightarrow p \end{array}$$

(ii) Given:

$$\begin{array}{l} p \Rightarrow q \\ \underline{\sim q \vee p} \\ p \Leftrightarrow q \end{array}$$

(iv) Given:

$$\begin{array}{l} a \vee \sim b \\ \underline{b} \\ a \end{array}$$

Partial Answers:

(ii) (1) $\left(\left(p \Rightarrow q\right) \wedge \left(p \vee q\right)\right) \Rightarrow \left(q \Rightarrow p\right)$

(3) Since there is an **F** in scenario 3 of column 7 (the final column), the implication is not a tautology. Thus, the argument is **invalid**.

(iii) (1) $\left(\left(p \Rightarrow q\right) \wedge \left(\sim q \vee p\right)\right) \Rightarrow \left(p \Leftrightarrow q\right)$

(3) Since every entry in column 8 (the final column) is **T**, the implication is a tautology. Thus the argument is **valid**.

(iv) (1) $\left(\left(a \vee \sim b\right) \wedge b\right) \Rightarrow a$

(3) The implication is a tautology, and the argument is **valid**.

XIX. SYLLOGISMS

Def. 15: **Syllogism –** A syllogism is a deductive argument with exactly two premises.

Many deductive arguments are syllogisms, and many, more extended deductive arguments can be "broken down" into a sequence of syllogisms.

(A) Some Valid Syllogisms

There are four valid syllogistic forms that occur frequently.

1. Modus ponens (MP)

The FORM	A SUBSTITUTION INSTANCE
$p \Rightarrow q$	If it rains, then Jones gets wet.
$\underline{p}$	It rains.
q	Jones gets wet.

Here is another substitution instance.

If you make a B on the final exam and turn-in your class project, you will not make below a C in the course.
You make a B on the final exam and you turn-in your class project.
You do not make below a C in the course.

Any substitution instance of a valid argument form is a valid argument.

Why is **MP** a valid form? We've already done this once, but I'll do it again.
Look at the implication: $((p \Rightarrow q) \wedge p) \Rightarrow q$ and its corresponding **TT**:

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \wedge p$	$((p \Rightarrow q) \wedge p) \Rightarrow q$
T	**T**	**T**	**T**	**T**
T	**F**	**F**	**F**	**T**
F	**T**	**T**	**F**	**T**
F	**F**	**T**	**F**	**T**
1	2	3	4	5

The **TT** shows that the implication is a tautology; therefore, the argument is **valid.**

2. Modus tollens (MT)

The FORM	A SUBSTITUTION INSTANCE
$p \Rightarrow q$	If it rains, then Jones gets wet.
$\sim q$	Jones does not get wet.
$\sim p$	It does not rain.

Another substitution instance:

If $a = 2$, then $a^2 = 4$.
$a^2 \neq 4$
$a \neq 2$

In paragraph form:

If $a = 2$, then $a^2 = 4$. But $a^2 \neq 4$, so $a \neq 2$.

MT is a valid syllogistic form. Each substitution instance of MT is a valid argument.

3. Pure Hypothetical Syllogism (HS)

The FORM	A SUBSTITUTION INSTANCE
$p \Rightarrow q$	If it rains, then Jones gets wet.
$q \Rightarrow r$	If Jones gets wet, then he will catch a cold.
$p \Rightarrow r$	If it rains, then Jones will catch a cold.

HS is a valid syllogistic form.

4. Disjunctive Syllogism (DS)

The FORM	A SUBSTITUTION INSTANCE
$p \vee q$	I'm going to study for the test or go to the concert.
$\sim p$	I'm not going to study for the test.
q	I'm going to the concert.

DS is a valid syllogistic form.

(B) Some Invalid Syllogisms There are two invalid syllogistic forms that occur frequently.

1. Affirming the Consequent (AC)

The FORM	A SUBSTITUTION INSTANCE
$p \Rightarrow q$	If it rains, then Jones gets wet.
q	Jones gets wet.
p	It rains.

Consider the implication: $((p \Rightarrow q) \wedge q) \Rightarrow p$

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \wedge q$	$((p \Rightarrow q) \wedge q) \Rightarrow p$
T	**T**	**T**	**T**	**T**
T	**F**	**F**	**F**	**T**
F	**T**	**T**	**T**	***F***
F	**F**	**T**	**F**	**T**
1	2	3	4	5

This implication is not a tautology, so the argument is **invalid**.

Example: There is a theorem in Calculus that states:

"If a function is differentiable, then it is continuous."[59]

Now suppose I am considering the function $g(x)=|x|$. I do know that $g(x)$ is continuous at x = 0. However, if I (incorrectly) use the theorem above to conclude that $g(x)$ is differentiable at x = 0, then I am guilty of the **fallacy**[60] **of affirming the consequent**.[61] In effect, I have reasoned thus:

$$\begin{array}{l} d \Rightarrow c \\ \underline{c \qquad} \\ d \end{array}$$

Not only is my **reasoning fallacious**, my conclusion is **false**!

Here is another exmple:

$$\begin{array}{l} \text{If a number is greater than zero, then the number is non-negative.} \\ \underline{\text{The number } a=0 \text{ is non-negative.} \qquad\qquad} \\ \text{The number } a=0 \text{ is greater than zero.} \end{array}$$

This example exhibits the hallmark characteristic of **invalid reasoning**, of a **formal fallacy**, namely, in this example ***both premises are true, yet the conclusion is false***.[62]

Reasoning which uses **AC** is always invalid. It should always be rejected.

Even if I *agree with the conclusion* of your argument by **AC**, I reject your *argument*. There is a significant difference between rejecting an *argument* and stating that a *conclusion is false*.

Another fallacious argument form is "Denying the Antecedent."

2. Denying the Antecedent (DA)

The FORM	A SUBSTITUTION INSTANCE
$p \Rightarrow q$	If it rains, then Jones gets wet.
$\underline{\sim p \quad}$	It does not rain.
$\sim q$	Jones does not get wet.

---

[59] You'll learn what "continuous" and "differentiable" are when you take Calculus. Suffice it to say at this time that "continuous" means that you can draw the graph without taking your pencil up from the paper, and "differentiable" means that at any point on the graph you can draw a line tangent to the graph.

[60] A **fallacy**, in this case a **formal fallacy** is the use of an **invalid argument** to draw a conclusion.

[61] The fallacy of **AC** is very common, even among folks who should know better!

[62] Recall that in our explanation of Def.13, we stated that a deductive argument is **valid** if and only if "it is impossible for the conclusion to be false whenever all the premises are true."

The form is **invalid**. Any argument based on this form is **invalid**.

As a summary of this section, let me say that if you are faced with an argument and you see that it is one of the four valid forms, then you know that it is valid. If it is one of the two invalid forms, then it is invalid. If it is "none of the above," then you have to make a truth table for the implication and determine on your own whether the argument is valid or invalid.

XX. PROBLEMS For each of the following symbolic arguments **(a)** Name the form, if it has a name. If it has no name, answer "No Name." **(b)** If it has a name, state the "initials" of the name. If it has no name, answer "NN." **(c)** State whether it is valid or invalid. If it has no name, you will have to support your answer to this part with an appropriate truth table.

1. $$\begin{array}{l} a \Rightarrow b \\ a \vee b \\ \hline a \wedge \sim b \end{array}$$

2. $$\begin{array}{l} a \Rightarrow b \\ a \vee b \\ \hline b \end{array}$$

3. $$\begin{array}{l} c \Rightarrow d \\ \sim c \\ \hline \sim d \end{array}$$

4. $$\begin{array}{l} (u \wedge a) \vee (a \Rightarrow u) \\ \sim (u \wedge a) \\ \hline a \Rightarrow u \end{array}$$

5. $$\begin{array}{l} p \Rightarrow (m \vee n) \\ \sim (m \vee n) \\ \hline \sim p \end{array}$$

6. $$\begin{array}{l} p \Rightarrow \sim t \\ \sim t \Rightarrow z \\ \hline p \Rightarrow z \end{array}$$

7. $$\begin{array}{l} m \Rightarrow n \\ m \wedge n \\ \hline n \end{array}$$

XXI. LOGICAL EQUIVALENCE

Def. XX: **Logically equivalent –** If two statements have identical truth tables (*i.e.,* the final columns are exactly the same), then we say that the two statements are **logically equivalent**.

We shall use the symbol "$::$" for logical equivalence. Thus "$p :: q$" is pronounced "*p* is logically equivalent to *q*."

This definition facilitates our stating the important **Law of Substitution**.

Law of Substitution. If $p :: q$ (where p and q may themselves be simple or compound statements), then if a compound statement contains p, the p may be replaced by q without changing the truth value of the compound statement.[63]

Example: It is easy to establish that $(p \vee q) :: (q \vee p)$.[64] Therefore $((q \vee p) \wedge \sim p) \Rightarrow q$ can be rewritten as $((p \vee q) \wedge \sim p) \Rightarrow q$ without changing the truth table of the re-written statement.

[63] One immediate consequence of the definition of logical equivalece is that $p :: q$ if and only if $p \Leftrightarrow q$ is a tautology.

[64] By this time it should be fairly obvious to you that the truth tables of $p \vee q$ and $q \vee p$ are the same.

Thus the argument   is logically equivalent to the argument

$$\begin{array}{l} q \vee p \\ \sim p \\ \hline q \end{array} \qquad\qquad \begin{array}{l} p \vee q \\ \sim p \\ \hline q \end{array}$$

But this latter argument is **DS**, which we know is valid. Therefore, the former argument is also valid.

XXI. RULES OF REPLACEMENT

There are ten important logical equivalences that have become to be known as the **rules of replacement**. Here they are:

#	NAME	SYMBOL	RULE (Logical Equivalence)
1	De Morgan's Laws	De M.	$\sim(p \wedge q) :: (\sim p \vee \sim q)$ $\sim(p \vee q) :: (\sim p \wedge \sim q)$
2	Commutation	Comm.	$(p \vee q) :: (q \vee p)$ $(p \wedge q) :: (q \wedge p)$
3	Association	Assoc.	$(p \vee (q \vee r)) :: ((p \vee q) \vee r)$ $(p \wedge (q \wedge r)) :: ((p \wedge q) \wedge r)$
4	Distribution	Dist.	$(p \wedge (q \vee r)) :: ((p \wedge q) \vee (p \wedge r))$ $(p \vee (q \wedge r)) :: ((p \vee q) \wedge (p \vee r))$
5	Double Negation	DN	$p :: \sim(\sim p)$
6	Transposition	Trans.	$(p \Rightarrow q) :: (\sim q \Rightarrow \sim p)$
7	Material Implication	Impl.	$(p \Rightarrow q) :: (\sim p \vee q)$
8	Material Equivalence	Equiv.	$(p \Leftrightarrow q) :: ((p \Rightarrow q) \wedge (q \Rightarrow p))$
9	Exportation	Exp.	$p :: (p \vee p)$
10	Tautology	Taut.	$p :: (p \wedge p)$

How can this information be used? Well, consider this argument:

$$\begin{array}{l} (p \wedge (q \vee r)) \\ \sim(p \wedge q) \\ \hline p \wedge r \end{array}$$

On first glance, this argument does not seem to fit into any one of the six "known" forms;[65] however, if we apply **Dist**. to the major premise, the argument is transformed into

[65] The four named valid forms and the two named invalid forms.

$$\begin{array}{l} ((p \wedge q) \vee (p \wedge r)) \\ \sim(p \wedge q) \\ \hline p \wedge r \end{array}$$

and this is seen to be **DS**, which is, of course, valid. Thus, the original argument is **valid**.

XXII. PROBLEMS Now we can look at slightly more complicated arguments and identify their forms.

For each of the following symbolic arguments **(a)** Name the form, if it has a name. If it has no name, answer "No Name." **(b)** If it has no name, state whether it can be re-written using a **rule of replacement** (and if so, which one) so that it "becomes" a named argument form, and state the name. **(c)** If it has a name (either originally or re-written), state the "initials" of the name. If it has no name, answer "NN." **(d)** State whether it is valid or invalid. If it has no name, you will have to support your answer to this part with an appropriate truth table.

1. $$\begin{array}{l} \sim s \Rightarrow \sim t \\ t \\ \hline s \end{array}$$

2. $$\begin{array}{l} \sim(a \wedge b) \\ b \\ \hline \sim a \end{array}$$

3. $$\begin{array}{l} p \vee (q \wedge s) \\ \sim q \vee \sim s \\ \hline \sim p \end{array}$$

4. $$\begin{array}{l} (a \wedge b) \vee c \\ \sim a \vee \sim b \\ \hline c \end{array}$$

5. $$\begin{array}{l} e \vee (f \vee g) \\ \sim(e \vee f) \\ \hline g \end{array}$$

6. $$\begin{array}{l} s \Leftrightarrow t \\ \sim s \\ \hline \sim t \end{array}$$

XXIII. VARIATIONS OF THE CONDITIONAL

(A) VARIATIONS IN WORDING.

The basic conditional statement is $p \Rightarrow q$. It can be translated into English in several ways. The most common is, of course,

If p, then q.

However, you will run into several variants of this wording, all of which mean $p \Rightarrow q$. Here are some of them:

q if p.
q whenever p.
p implies q.
p only if q.[66]

[66] Think of this last re-wording as "If q doesn't happen, then p doesn't happen." And this is an example of **transposition**, which was discussed in a previous section.

Thus,

$$\left| f(x) - L \right| < \varepsilon \quad \text{whenever} \quad 0 < \left| x - a \right| < \delta$$

is just an alternate way of writing

$$If \quad \left| x - a \right| < \delta, \quad then \ \left| f(x) - L \right| < \varepsilon.$$ [67]

<u>(B) STATEMENTS DERIVED FROM AND/OR RELATED TO THE CONDITIONAL.</u>

1. Let's start with a conditional statement (implication) $p \Rightarrow q$.
2. We can form $q \Rightarrow p$ (the **converse of the conditional**).
3. Or we can form $\sim q \Rightarrow \sim p$ (the **contrapositive of the conditional**).
4. And we can from $\sim p \Rightarrow \sim q$ (the **inverse of the conditional**).

As it turns out,

- the original conditional statement and its contrapositive are logically equivalent;
- the converse and the inverse are logically equivalent;
- but the original conditional statement and its converse are not logically equivalent.

We shall establish the first and third of these "bullets."

Recall that two statements are logically equivalent if and only if their truth tables are identical in the final columns.

<u>(1) Consider a conditional statement and its contrapositive:</u>

p	q	$p \Rightarrow q$	$\sim p$	$\sim q$	$\sim q \Rightarrow \sim p$
T	**T**	**T**	**F**	**F**	**T**
T	**F**	**F**	**F**	**T**	**F**
F	**T**	**T**	**T**	**F**	**T**
F	**F**	**T**	**T**	**T**	**T**
1	2	3	4	5	6

Here we have combined two truth tables into one panel. Columns 1, 2, and 3 are the truth table for the conditional statement, $p \Rightarrow q$, and columns 1, 2, 4, 5, and 6 constitute the truth table for the contrapositive, $\sim q \Rightarrow \sim p$.

[67] This implication is ***<u>extremely important</u>*** in the Calculus.

Now, you can see that column 3 and column 6 are identical. Therefore, ***the conditional and its contrapositive are logically equivalent.***

(2) Consider a conditional statement and its converse:

p	q	$p \Rightarrow q$	$q \Rightarrow p$
T	**T**	**T**	**T**
T	**F**	**F**	**T**
F	**T**	**T**	**F**
F	**F**	**T**	**T**
1	2	3	4

In this combined truth table, columns 1, 2, and 3 give us the truth table for the conditional statement, and columns 1, 2, and 4 give us the truth table for the converse of the conditional statement.

You can see that these two truth tables are not the same; therefore, the two statements are not logically equivalent.

So just what is the lesson to be learned here? It is to never confuse a conditional and its converse, because they mean different things.

XXIV. PROOFS

Suppose that I want to prove the following:

$$If \quad k \in \mathbb{Z},\ k > 2 \ \ and \ \ 2k+1 < 2^k,\ then \ \ 2k+3 < 2^{k+1}.$$

Many people would (incorrectly) try to prove this little theorem by *enumeration*. They would “check” the final inequality. That is they would say something like this:

$$\begin{aligned}
&k \ \ must \ be \ an \ \text{integer} \ greater \ than \ 2, \ so \ start \ with \ k=3: \\
&if \ \ k=3, \ then \ 2(3)+3=9 \ and \ 2^4=16, \ so \ 2k+3<2^{k+1}. \\
&If \ \ k=4, \ then \ 2(4)+3=11 \ and \ 2^5=32, \ so \ 2k+3<2^{k+1}. \\
&If \ \ k=5, \ then \ 2(5)+3=13 \ and \ 2^6=64, \ so \ 2k+3<2^{k+1}. \\
&If \ \ k=6, \ then \ 2(6)+3=15 \ and \ 2^7=128, \ so \ 2k+3<2^{k+1}. \\
&etc. \\
&Thus, \ it \ follows \ that \ whenever \ k>2, \ 2k+3<2^{k+1}.
\end{aligned}$$

There are quite a few things wrong with this "proof." Can you find any errors in this "proof?" I can find some. Let me list my comments:

- The "prover" (one who has proved) has demonstrated that the inequality $2k+3<2^{k+1}$ holds for $k=3,\ 4,\ 5,\ and\ \ 6$, but the prover hasn't demonstrated that the inequality holds for, say $k=17,\ 29,\ 356,\ and\ \ 86400$, to name just a few more positive integers. What's my point here? The theorem requires that the inequality be true for ***<u>all</u> integers greater than 2, not just <u>some</u> integers greater than 2***.

- Although the theorem in question is true[68], the same (flawed) method of proof might be used to "prove" the false statement
$$If\ \ n\in\mathbb{Z}^+,\ \ then\ \ n^2-n+11\ \ is\ \ a\ \ prime\ \ number.$$
You try the "proof by inadequate enumeration" on this statement. You'll see that the conclusion is true for $n=1,\ 2,\ 3,\ 4,\ 5,\ 6,\ and\ \ 7$ (isn't that enough?). But wait! It's also true for $n=8,\ 9,\ and\ \ 10$. Wow! But, woops, as you could see if you had looked carefully at the polynomial,
$$if\ \ n=11,\ then\ \ n^2-n+11=11^2-11+11=11^2=121,$$
which is <u>not</u> a prime number. Thus, if I use this technique of "proof," how do I know whether I am proving a true statement true or a false statement true? The method is flawed. I reject it.

- Another consideration that I have about the alleged proof is that not all the hypotheses (premises) were used in deriving the conclusion. As you can see, the "proof" makes no use of the hypothesis that $2k+1<2^k$. This omission makes me very suspicious that something is amiss in the "proof."

OK, so how should a legitimate proof go? I'm going to be really formal here.

- A proof consists of a sequence of steps.
- Each step has three parts –
  - a step number,
  - the step itself, and
  - a reason for the step.
- Each step consists of
  - a premise (hypothesis),
  - a mathematical "fact of life" (an axiom, postulate, or previously proven theorem), or
  - a logical conclusion derived from previous steps in the proof.
- The steps form a path which leads ultimately to the conclusion of the theorem.

Once you get good at proofs, you might shorten the process somewhat.

- Sometimes you won't write-out the reasons for each step
- (but you should always be able to silently state each reason to yourself, and if you can't give a reason, don't take the step)!
- Some teachers just give the steps in writing the proof on the board, but they usually verbally state the reason, if it is not obvious.

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[68] I've always had a problem with the phrase "true theorem," because I'm not sure if there is such a thing as a "false theorem." If it's false, then can it really be a theorem? Well, let's let that pass and move on.

- I have done some engineering in my life, and I have found that engineers like to break things down into steps, so I usually list my step number and then write the step.
- Again, I don't always write-out the reason, unless it is not obvious to my audience. Thus, I write more reasons in my Intermediate Algebra classes than I do in my Differential Equations classes.
- But in all cases I do try to list all the steps necessary for the student to follow the proof, both at the moment and later during review of the notes.
- On the other hand, the style of textbook writing calls for omitting steps (but not the most difficult ones). Thus, when reading a textbook, you will need a pencil or pen so that you can re-create the omitted steps. I guess that's why they put in those ample margins. (Some of the most expensive note paper this side of the DOD (Department of Defense)).

Well, I guess I'm finally going to get around to proving the theorem. I'll do it twice. First you get the "long version." Then you get the "short version."

Theorem: *If* $k \in \mathbb{Z},\ k > 2$ *and* $2k+1 < 2^k$, *then* $2k+3 < 2^{k+1}$.

Proof:

Step #	Step	Reason
1	$2k+1 < 2^k$	Given
2	$2k+3 = 2k+1+2 = (2k+1)+2$	Simple algebra. My goal: Trying to get the LHS (left hand side) of the conclusion to somewhat resemble my "given."
3	$2k+3 < 2^k + 2$	Combining steps 2 & 1. (or Add 2 to both sides of step 1.)
4	$(k>2) \Rightarrow (2 < 2^k)$	$k > 2$ is given. And any positive power of 2 is greater than 2.
5	$2k+3 \ < \ 2^k+2 \ < \ 2^k+2^k$	Combining steps 3 & 4.
6	$2k+3 \ < \ 2^k+2^k$	Rewriting step 5 with the "middle inequality" left out. For clarity. OR using the transitivity property of inequality: $((a<b) \wedge (b<c)) \Rightarrow (a<c)$
7	$2^k+2^k = 2\times 2^k$	"One apple plus one apple equals two apples." This is really the key step. That's why I'm isolating it here.
8	$2\times 2^k = 2^{k+1}$	Rules of exponents.
9	$2k+3 \ < \ 2^{k+1}$	This is the conclusion. This is my "donut.[69]" I combined steps 6, 7 & 8.

I wrote-out this proof in this detail just to prove a point – to show you that I had a reason for everything I did.

[69] "As you wander through life,
Let this be your goal.
Keep your eye on the donut,
And not on the hole."

If I were to write-out the proof again at the end of the semester, when you might be a little more comfortable with the process, my proof might look like this:

Theorem: $If \quad k \in \mathbb{Z},\ k > 2 \quad and \quad 2k+1 < 2^k,\ then \quad 2k+3 < 2^{k+1}.$

Proof:

$k > 2$ (given), so $2 < 2^k$.

And $2k+1 < 2^k$ (given).

Thus, $2k+3 = (2k+1)+2 < 2^k + 2 < 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$.

$\therefore \ 2k+3 < 2^{k+1}$

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